EXISTENCE OF X-PATHS IN ABSTRACT POLYTOPES

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ILAN ADLER, GEORGE DANTZIG, KATTA MURTY

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Given a finite set E of n symbols ($n \geq 2$), a family S of subsets of E (called vertices) form an <u>abstract polytope</u> (E,S) if the following three axioms are satisfied.

[i] Each vertex is a subset of m symbols of E ($m \leq n$).

[ii] Every subset of $m + 1$ symbols contains either zero or two vertices.

Before stating the third axiom let us introduce two definitions.

<u>Adjacency</u> - Two vertices V^0 and V^* are said to be <u>adjacent</u> (or <u>neighbors</u>) if $V^0 \cup V^*$ has cardinality $m + 1$.

<u>Path</u> - A sequence of vertices $V_1,\ldots,V_k$ is called a path if V_i, V_{i+1} are adjacent for $i = 1,\ldots,k-1$.

[iii] Every pair of vertices V^0 and V^* can be joined by a path $V^0 = V_1,\ldots,V_k = V^*$ such that $V_i \subset V^0 \cup V^*$ $i = 1,\ldots,k$.

The <u>dimension</u> of (E,S) is defined as $n - m$, and the <u>Graph</u> of (E,S) is defined as the graph which is formed from the vertices of S and the arcs which join adjacent vertices.

Note that by axioms [i] and [ii] every vertex in (E,S) has exactly $n - m$ neighbors.

__Theorem__ Given an abstract polytope (E,S) if two vertices V^0 and V^* in S each contain the same symbol (say x) there is an x-path joining them, i.e., there exists a path such that every vertex along the path contains x.

Abstract polytopes include as special cases simple convex polytopes. The latter can be represented as non-degenerate bounded feasible linear programs. The analoguous theorem states: If two feasible bases have a column in common, then it is possible to pass from one to the other via a sequence of adjacent feasible basis changes all of which have the column in common. The usual proof is to assign as linear objective the maximization of the variable of the common column. Then two paths exist one from each of the initial feasible bases to an optimal basis. All bases along each path have the column in common. Hence if there is a unique optimal basis, the paths may be joined together to form a simple connecting path. It is not hard to extend the result to the non-unique case as well. For abstract polytopes, however, there are no basic variables nor an objective function, hence a different proof is required.

If Γ is any subset of E, we denote by card (Γ), the cardinaltiy of Γ.

__Proof__ __Case 1__ - card $(V^0 \cup V^*) = m + 1$.

In this case V^0 and V^* are adjacent so V^0, V^* is an x-path joining V^0 and V^*.

__Case 2__ - card $(V^0 \cup V^*) = m + 2$.

Let $E^1 = V^0 \cup V^*$ and $S^1 = \{V \in S | V \subset E^1\}$. By axioms [i] -

[iii], (E^1,S^1) is a 2-dimensional abstract polytope. Hence the graph G of (E^1,S^1) forms a simple cycle. Thus there are two separate paths joining V^0 to V^*. Suppose the theorem is false, i.e., in each of the two paths there exists a vertex (say V^1 and V^{11}) such that $x \notin V^1$ and $x \notin V^{11}$. But then card $(V^1 \cup V^{11}) \leq m + 1$, so either $V^1 = V^{11}$ or V^1 is adjacent to V^{11}. Both cases contradict the cycle structure of G.

Therefore, at least one of the two paths joining V^0 and V^* is an x-path.

Case 3 - card $(V^0 \cup V^*) \geq m + 3$.

Let $F = E - x$ and $R = \{V \in S | V \subset F\}$. If $R = \emptyset$ then obviously there exists an x-path joining V^0 and V^* (by axiom [iii]). If $R \neq \emptyset$ then (F,R) is n-m-1 dimensional abstract polytope. By axiom [iii] there exists a path $V^0 = V_1,\ldots,V_k = V^*$ in S. If $V_i \notin R$ for all $i = 1,\ldots,k$ then $V_1,\ldots,V_k$ is an x-path joining V^0 and V^*. If not, let f and ℓ be, respectively, the first and last index i such that $V_i \in R$. Since (F,R) is an abstract polytope there exists a path $V_f = R_1,\ldots,R_p = V_\ell$ such that $R_i \in R$ $i = 1,\ldots,p$. Let $S_1,\ldots,S_p$ be respectively, the neighbors of $R_1,\ldots,R_p$ in $S-R$. (Note that every vertex in R has a unique neighbor in $S-R$, while a vertex in $S-R$ can be a neighbor of zero or several vertices in R.)

Since R_i, R_{i+1} $(i = 1,\ldots,p-1)$ differ by exactly one symbol it follows that the adjacent S_i and S_{i+1} (formed by adjoining the

common symbol x to R_i and R_{i+1}) differ by at most two symbols. Thus by cases 1 and 2, S_i and S_{i+1} $(i = 1,\dots,p-1)$ can be joined by an x-path. So there is an x-path joining S_1 and S_p; let us denote this x-path by $S_1 = \bar{S}_1,\dots,\bar{S}_q = S_p$.

It follows that $V^0 = V^1,\dots,V_{\ell-2}, \bar{S}_1,\dots,\bar{S}_q, V_{\ell+2},\dots,V_k = V^*$ is an x-path joining V^0 and V^*.

Q.E.D.

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→ Given a finite set E of n symbols a family S of subsets of E (called vertices) form an abstract polytope if

1) Each vertex is a subset of m symbols of E.

2) Every subset of $m+1$ symbols of E contains either zero or two vertices (called adjacent). →V sup 0 and V sup *

3) Every pair of vertices V^0 and V^* can be joined by a path $V^0 = V_1, \ldots, V_k = V^*$ such that V_i, V_{i+1} are adjacent and $V_i \subset V^0 \cup V^*$ $i = 1, \ldots, k-1$

It is shown that if two vertices of a given abstract polytope contain the same symbol (say x) then there exists a path such that every vertex along the path contains x. (x)

→ V sub i, V sub (i+1)

→ (V sup 0 = V sub 1, ..., V sub k = V sup *

→ (V sub i) contained in (V sup 0) joined to (V sup *)

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